

Linear Programming

Background Information on VBA Programming in Business Economics by Sanne Wøhlk

Linear programming problems (LP) are problems that can be described by a set of linear constraints of the type $a_1 x_1 + a_2 x_2 + a_3 x_3 + \dots + a_n x_n \leq b$ where $b \in \mathbb{R}$, $a_i \in \mathbb{R}$ for all i , and where $x_i, i = 1, \dots, n$ are variables. Furthermore, a linear objective function is defined as $c_1 x_1 + c_2 x_2 + c_3 x_3 + \dots + c_n x_n$ where $c_i \in \mathbb{R}$ for all i . The goal is to determine the values of the variables that either minimize or maximize the value of the objective function depending on the nature of the problem. Hence, a linear programming problem is a problem that can be described as

$$\begin{array}{rcccccccc}
 \text{max} & c_1 x_1 & + c_2 x_2 & + c_3 x_3 & \dots & + c_n x_n & & \\
 \text{s. t.} & a_{11} x_1 & + a_{12} x_2 & + a_{13} x_3 & \dots & + a_{1n} x_n & \leq & b_1 \\
 & a_{21} x_1 & + a_{22} x_2 & + a_{23} x_3 & \dots & + a_{2n} x_n & \leq & b_2 \\
 & a_{31} x_1 & + a_{32} x_2 & + a_{33} x_3 & \dots & + a_{3n} x_n & \leq & b_3 \\
 & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
 & a_{m1} x_1 & + a_{m2} x_2 & + a_{m3} x_3 & \dots & + a_{mn} x_n & \leq & b_m
 \end{array}$$

In the model, *max* can be replaced by *min*. This problem can be presented more compactly using matrix notation. Let c and x be vectors of dimension n , b be a vector of dimension m , and let A be an n times m matrix. Then the problem can be formulated as

$$\begin{array}{l}
 \text{max} \quad c^T x \\
 \text{s. t.} \quad Ax \leq b
 \end{array}$$

This is the standard form of a linear programming problem. Constraints of the form \geq can easily be included by multiplication by -1 and constraints of $=$ form can be included using a pair of inequality constraints. Note that for practical purposes it is not necessary to convert constraints to standard form as all standard software can handle constraints of the forms $\leq, =$, and \geq .

Often the variables are required to be non-negative, in which case the problem can be stated as follows where the last constraint is referred to as the non-negativity constraint.

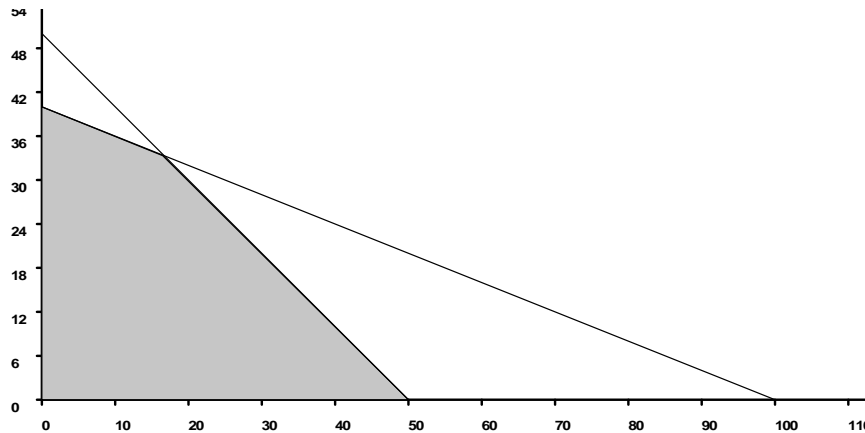
$$\begin{array}{l}
 \text{max} \quad c^T x \\
 \text{s. t.} \quad Ax \leq b \\
 \quad \quad x \geq 0
 \end{array}$$

As an example, consider a farmer who has 50 acres of land where he can grow two products: Wheat and corn. He wants to determine the amount of each to plant during the coming season in order to maximize his profit. Each acre of wheat will give him a profit of 40\$ and requires 0.2 hours of work per week. Each acre of corn will give him a profit of 60\$ and requires 0.5 hours of work per week. He has 20 hours per week available for work with crops.

To formulate this problem as a linear programming problem, let x_1 be the number of acres used to grow wheat and let x_2 be the number of acres used to grow corn. The total profit obtained by the farmer is given by $40 x_1 + 60 x_2$. The availability of land states that $x_1 + x_2$ can be no more than 50 and the limitations of working hours states that $0.2 x_1 + 0.5 x_2$ may not exceed 20. The complete problem is defined by the following linear programming model:

$$\begin{aligned}
\max \quad & 40x_1 + 60x_2 \\
\text{s. t.} \quad & x_1 + x_2 \leq 50 \\
& 0.2x_1 + 0.5x_2 \leq 20 \\
& x_1, x_2 \geq 0
\end{aligned}$$

Because the problem only includes two variables the feasible region of the problem, i.e. the set of values of the variables that respects all constraints, can be illustrated as the gray area below. The goal of the problem is to select a point within this region which maximizes the objective function. Note that such a point can always be found on the boundary of the region.



Integer linear programming problems (ILP) are problems that can be described as an LP with the additional requirement that all variables must have integer values. Mixed integer linear programming problems (MILP) are problems that can be described as an LP with the additional requirement that some, but not all, variables must have integer values. In practice MILPs are often referred to as ILPs.

As an example of an ILP consider a customer who wants to buy each of the products A, B, ..., F. The products cannot be bought separately, but are packed in sets. The available sets are given below along with their price. Note that only whole sets can be bought. The goal of the customer is to buy sets in the cheapest possible way to obtain at least one unit of each product

Set	Products	Price
1	A, B, C	105
2	A, C, D, E	158
3	A, E, F	172
4	B, C, D, E	207
5	B, C, E, F	225
6	B, C, F	163
7	B, E, F	199
8	C, D, E	168

To formulate this problem as an ILP we let each set correspond to a variable. The value of the variable states the units of the set to be bought. The objective coefficients correspond to the price of the sets. A \geq constraint is imposed for each product with the right hand side being 1, corresponding to one unit being wanted. In the constraint matrix, a_{ji} is 1 if product j is contained in set i and 0 otherwise. The model for the problem is given below.

